INTERACTION BETWEEN A CYLINDRICAL WAVE IN WATER AND THE FREE SURFACE

B. V. Boshenyatov and B. I. Zaslavskii

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, no. 5, pp. 95-98, 1968

The problem of the reflection of a cylindrical shock wave in water from the free surface is analyzed within the framework of shortwave theory [1, 2]. The motion is studied essentially on the basis of results obtained in [2-4].

1. Let a shock wave be generated by an explosion of an infinite cylindrical charge, per unit length of which a certain energy, characterized by the linear dimension R_0 (the charge radius), is released. The axis of the charge is parallel to the free surface at a certain depth h. We align the axis of the cylindrical system of coordinates r, θ with the charge axis. The angle θ will be laid off from a plane parallel to the free surface. The flow behind the shock wave front will be determined on the basis of the system of "shortwave" equations [1, 2]

$$\frac{\partial\mu}{\partial\tau}+(\mu-\delta)\frac{\partial\mu}{\partial\delta}+\frac{1}{2}\frac{\partial\nu}{\partial y}+\frac{1}{2}\mu=0,$$
$$\frac{\partial\nu}{\partial\delta}-\frac{\partial\mu}{\partial y}=0. \qquad (1.1)$$

The quantities μ, ν, y, δ are defined by the equalities

$$U=a_0M_0\mu, \qquad V=a_0M_0\sqrt{\frac{n+1}{2}M_0}\ \nu,$$
$$r=a_0t\left(1+\frac{n+1}{2}M_0\delta\right), \qquad \tau=\ln\frac{a_0t}{R_0},$$
$$0=y\sqrt{\frac{n+1}{2}M_0}, \quad M_0\mu=\frac{p}{Bn}, \quad R=\frac{r}{R_0},$$
$$b=3045\ \text{atm}, \quad n=7.15,$$

where a_0 is the initial speed of sound; U and V are the projections of the particle velocity vector on the direction of radius vector, and on the direction normal to it, respectively; M_0 is a certain small quantity; and R is a distance expressed in terms of the charge radius R_0.

From (1.1) it is possible to obtain the flow which forms behind the shock wave front in an unbounded fluid, in which case $\nu = 0$, i.e., the flow is one dimensional [2]. We get

$$\delta=2\mu+e^{-\tau}\Phi(\mu^2e^{\tau}), \qquad (1.2)$$

where Φ is an arbitrary function. In the following, we assume that the pressure profile behind the shock front is triangular; then

$$\delta=2\mu+Ce^{-\tau}. \qquad (1.3)$$

The position of the shock front is defined by the equation

$$\delta+\frac{\partial\delta}{\partial\tau}=\frac{1}{2}\mu+\frac{1}{2}\left(\frac{\partial\delta}{\partial y}\right)^2. \qquad (1.4)$$

In the one-dimensional case, from (1.4) and (1.3), at the shock front, we get

$$\mu=Ae^{-3\tau/4}, \qquad \delta=2Ae^{-3\tau/4}+Ce^{-\tau} \qquad (A,\ C=\text{const}). \quad (1.5)$$

From (1.5) and (1.1), there follows Landau's [5] asymptotic formula

$$P=kR^{-3/4} \qquad (k=\text{const}). \qquad (1.6)$$

2. After the shock wave reaches the free surface, the rarefaction waves which propagate from the points of the free surface through which the shock front has passed interact with the flow (1.3). In the determination of the flow in the interaction region (which we term the disturbed flow), it is convenient to place the origin O of the system of coordinates on the free surface, at the point O_1 of intersection with the normal to the center of the charge (Figure 1). The transformation formulas from μ, ν, y, δ to μ_1, ν_1, y_1, δ_1 in the new system of coordinates have the form

$$\delta={}^{2\tau}+y_1y_0e^{-\tau}, \quad \nu_1=\nu-\mu y_0e^{-\tau},$$
$$y_1=y+y_0e^{-\tau}, \quad \mu_1=\mu$$
$$\left(y_0=\frac{h}{R_0\sqrt{{}^1/_2(n+1)M_0}}\right). \qquad (2.1)$$

We perform the following change of variables and functions:

$$\delta_1=[\delta^\circ+\delta_0(\tau)]e^{-{}^1/_2\tau}, \qquad y_1=y^\circ e^{-\tau/4},$$
$$\mu_1=[\mu^\circ+\mu_0(\tau)]e^{-{}^1/_2\tau}, \quad \nu_1=(\nu^\circ+2y^\circ\frac{d\mu_0}{d\tau})e^{-{}^3/_4\tau}. \qquad (2.2)$$

Here, $\delta_0(\tau)$ and $\mu_0(\tau)$ are unknown functions of time, $\delta_0e^{-\tau/2}$ is the value of δ_1 at point A of intersection of shock wave front and free surface, and $\mu_0e^{-\tau/2}$ is the value of μ_1 and at this point. After performing transformations (2.2), Eqs. (1.1) take the form

$$\frac{\partial\mu^\circ}{\partial\tau}+\left(\mu^\circ+\mu_0-\frac{\delta^\circ+\delta_0}{2}-\frac{d\delta_0}{d\tau}\right)\frac{\partial\mu^\circ}{\partial\delta^\circ}+$$
$$\frac{1}{4}y^\circ\frac{\partial\mu^\circ}{\partial y^\circ}+\frac{1}{2}\frac{\partial\nu^\circ}{\partial y^\circ}=0, \qquad \frac{\partial\mu^\circ}{\partial y^\circ}-\frac{\partial\nu^\circ}{\partial\delta^\circ}=0. \qquad (2.3)$$

3. The reflection of a shock wave from the free surface can involve regular and irregular reflections [3]. An irregular reflection starts from the instant the rarefaction waves propagating from the free surface catch up with the shock front. Assume that this instant is reached at a distance $R = R_*$ or correspondingly, $\tau = \tau_*$. At the shock wave front, we set

$$\mu_0(\tau_*)+\mu^\circ=1. \qquad (3.1)$$

The value of y_0 and M_0 can be expressed through τ_* [4],

$$y_0=e^{{}^3/_4\tau_*}, \qquad M_0=\frac{k}{Bn}e^{-{}^1/_4\tau_*}. \qquad (3.2)$$

Formulas (1.1), (2.1). (1.6) are used in the derivation of (3.2).

In the new system of coordinates, Eqs. (1.3) and (1.5) take the form

$$\delta^\circ=2(\mu_0+\mu^\circ)-{}^1/_2x^{-3}+y^\circ x^{-3/2}-{}^3/_2x^{-1}-\delta_0(\tau), \qquad (3.3)$$
$$\delta^\circ+\delta_0=2x^{-1/2}-{}^3/_2x^{-1}+y^\circ x^{-3/2}-{}^1/_2x^{-3}$$
$$(\mu^\circ+\mu_0=x^{-1/2}, \quad x=e^{{}^1/_2(\tau-\tau_*)}). \qquad (3.4)$$

The perturbed-flow region contacts flow (3.3) along the characteristic surface of Eqs. (2.3) (BD in Fig. 1). The equation of the characteristic surface of Eqs. (2.3) has the form

$$-\frac{\partial\delta^\circ}{\partial\tau}+\left(\mu^\circ+\mu_0-\frac{1}{2}\delta^\circ-\frac{1}{2}\delta_0-\frac{d\delta_0}{d\tau}\right)-$$
$$\frac{1}{4}y^\circ\frac{\partial\delta^\circ}{\partial y^\circ}+\frac{1}{2}\left(\frac{\partial\delta^\circ}{\partial y^\circ}\right)^2=0. \qquad (3.5)$$

By integrating (3.5), under the condition that the required surface extend along (3.3) and that it pass through point A at time $\tau = \tau_*$, we get

$$\delta^\circ=q_0(\tau)y^{\circ 2}+\chi_1{}^\nu(\tau)y^\circ+\chi_0{}^\nu(\tau), \qquad (3.6)$$

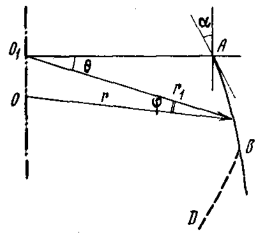

Fig. 1

$$q_0 = -{}^1/_4\ (x-1)^{-1}, \quad \chi_1^{\nu} = -{}^1/_2 x^{-3/2}(x-1),$$
$$\chi_0^{\nu} = -{}^1/_4 x^{-3} + {}^1/_4 x^{-2} - {}^7/_4 x^{-1} + {}^7/_4 - \delta_0(\tau). \qquad (3.7)$$

The point of intersection B (Fig. 1) of the characteristic surface BD and the shock front separates the segment of the front AB which is distorted by the influence of the free surface from the undistorted portion of the shock front. The coordinates of this point are determined from (3.4) and (3.6),

$$y_B^{\circ} = -x^{-3/2}(x^2-1) +$$
$$+ [x^{-1}(x^2-1) - 4(x-1)(2x^{-1/2} + {}^1/_4\ x^{-1} - {}^7/_4)]^{1/2},$$
$$\delta_B^{\circ} = q_0 y_B^{\circ 2} + \chi_1^{\nu} y_B^{0} + \chi_0^{\nu}. \qquad (3.8)$$

In this way, the solution is sought in a region bounded by the free surface, the segment of the shock front AB and the characteristic surface BD. Near the point of intersection (A) of the shock front and the free surface, there occurs a Prandtl Meyer type flow [3], i.e., for $y^0 \to 0$, $\delta \to 0$,

$$\mu^{\circ} = -\frac{1}{2}\left(\frac{\delta^{\circ}}{y^{\circ}}\right)^2 + \mu_0(\tau), \qquad \nu^{\circ} = \frac{1}{3}\left(\frac{\delta^{\circ}}{y^{\circ}}\right)^3 + \nu_0(\tau). \quad (3.9)$$

Formulas (3.9) are the boundary conditions at the free surface. At the shock front, whose equation in δ^0, y^0, τ coordinates is of the form

$$\frac{\delta_0 + \delta^{\circ}}{2} + \frac{\partial \delta^{\circ}}{\partial \tau} + \frac{d\delta_0}{d\tau} + \frac{1}{4} y^{\circ} \frac{\partial \delta^{\circ}}{\partial y^{\circ}} =$$
$$\frac{1}{2}(\mu_0 + \mu^{\circ}) + \frac{1}{2}\left(\frac{\partial \delta^{\circ}}{\partial y^{\circ}}\right)^2, \qquad (3.10)$$

it is necessary to fulfill the condition for the continuity of the velocity vector component tangential to the shock front

$$(\mu^{\circ} + \mu_0)\frac{\partial \delta^{\circ}}{\partial y^{\circ}} + \nu^{\circ} = 0. \qquad (3.11)$$

For irregular reflection, at point A, the angle of incidence of the shock wave α (i.e., the angle formed by the shock front and the normal to the free surface) always retains its critical value [3, 4], i.e.,

$$\alpha = \left(\frac{n+1}{2}\frac{p'}{Bn}\right)^{1/2} = \alpha_*^{\circ}\left(\frac{n+1}{2}M_0\right)^{1/2} e^{-1/4\tau}, \qquad (3.12)$$

where p' is the pressure at the shock front at point A. From here, in δ^0, y^0, τ coordinates, we get

$$\alpha_*^{\circ} = \frac{\partial \delta^{\circ}}{\partial y^{\circ}}\bigg|_{y=0} = \sqrt{\mu_0(\tau)}, \qquad (3.13)$$

and, consequently,

$$\mu_0 = \frac{d\delta_0}{d\tau} + \frac{1}{2}\delta_0. \qquad (3.14)$$

At the characteristic surface ED, the flow to be determined must contact (3.3).

4. In order to determine the flow in region ABD, we use the exact particular solutions of system (2.3), obtained in [4]:

$$\mu^{\circ} = \varphi_2 y^{\circ 2} + \varphi_1 y^{\circ} + \varphi_0,$$
$$\nu^{\circ} = \psi_3 y^{\circ 3} + \psi_2 y^{\circ 2} + \psi_1 y^{\circ} + \psi_0,$$
$$\delta^{\circ} = q y^{\circ 2} + \chi_1 y^{\circ} + \chi_0, \qquad \varphi_2 = -{}^1/_2 q^2 + {}^1/_2 q + z(\tau),$$
$$\psi_3 = -{}^2/_3[\varphi_{2\tau} + \varphi_{2q}\ (\varphi_2 - q + 2q^2) + {}^1/_2\ \varphi_2 - 2\varphi_2 q],$$
$$\psi_2 = -\varphi_{1\tau} - \varphi_{1q}\ (\varphi_2 - q + 2q^2) + \chi_1\varphi_2 + \varphi_1 q - {}^1/_4\ \varphi_1,$$
$$\psi_1 = -2\varphi_{0\tau} - 2\ \varphi_{0q}(\varphi_2 - q + 2\ q^2) + \varphi_1\chi_1,$$
$$\psi_0 = \varphi_1\chi_{0q} - \chi_1\varphi_{0q},$$
$$\chi_1 = \int_{-\infty}\left(\int_{-\infty}^{q} w\,dq\right)dq, \quad \chi_0 = \int_{-\infty}^{q} v\,dq,$$
$$\varphi_1 = \int_{-\infty}^{q}\varphi_{2q}\left(\int_{-\infty} w\,dq\right)dq, \quad \varphi_0 = \int_{-\infty}^{q}\varphi_{2q}\, v\,dq. \qquad (4.1)$$

The functions w and v are determined from the equations

$$w_\tau + w_q({}^3/_2 q^2 - {}^1/_2 q + z) + w(5q - {}^3/_4) = 0,$$
$$v_\tau + v_q({}^3/_2 q^2 - {}^1/_2 q + z) + v(4q - {}^1/_2) - \chi_1\chi_{1q} = 0. \qquad (4.2)$$

Solutions (4.1) satisfy (3.9) for any finite values of $z(\tau)$, $v(q,\tau)$, and $w(q,\tau)$. We select these functions from the condition for the flow (4.1) to contact flow (3.3) at characteristic surface BD, at which $q = q_0(\tau)$.

We write (3.3) in the following way:

$$\mu^{\circ} = {}^1/_2\ q y^{\circ 2} + \varphi_1^{\nu} y^{\circ} + \varphi_0^{\nu}, \quad \delta^{\circ} = q y^{\circ 2} + \chi_1^{\nu} y^{\circ} + \chi_0^{\nu}, \qquad (4.3)$$
$$\nu = -(\mu^{\circ} + \mu_0)x^{-3/2} + 2\mu_0' e^{3/4\tau}.$$

$$\varphi_1^{\nu} = -{}^1/_4 x^{-3/2}(x+1),$$
$$\varphi_0^{\nu} = {}^1/_8 x^{-3} + {}^1/_8 x^{-2} + {}^1/_8 x^{-1} + {}^7/_8 - \mu_0(\tau). \qquad (4.4)$$

In order that (4.1) convert to (4.3) at characteristic curve BD the functions must satisfy the following conditions:

$$\varphi_2(q_0,\tau) = {}^1/_2 q_0, \qquad z(\tau) = {}^1/_2 q_0^2,$$
$$\chi_1(q_0,\tau) = \chi_1^{\nu}, \qquad \chi_0(q_0,\tau) = \chi_0^{\nu}, \qquad (4.5)$$
$$\varphi_1(q_0,\tau) = \varphi_1^{\nu}, \quad \varphi_0(q_0,\tau) = \varphi_0^{\nu},$$
$$\psi_0(q_0\tau) = -(\varphi_0^{\nu} + \mu_0)x^{-3/2},$$
$$\psi_1(q_0,\tau) = 2\mu_0' - \varphi_1^{\nu} x^{-3/2},$$
$$\psi_2(q_0,\tau) = -{}^1/_2 q_0 x^{-3/2}, \quad \psi_3(q_0,\tau) = 0. \qquad (4.6)$$

Fulfillment of (4.5) means also the fulfillment of (4.6). In this way, the problem reduces to the integration of the two equations (4.2) for condition (4.5). The general solution of these equations has the form

$$w = F_1(\eta)x^{3/2}\omega^{5/3}(\omega - 3\eta)^{10/3},$$
$$v = x\omega^{4/3}(\omega - 3\eta)^{8/3} \times$$
$$\times [\omega^4 + 8\eta\ \omega^3 - 26\eta^2\omega^2 - 168\ \eta^3\omega + F_2(\eta)],$$
$$\eta = \omega\frac{1 + 4(x-1)q}{1 + 12(x-1)q}, \quad \omega = \sqrt{\frac{x-1}{x}}, \qquad (4.7)$$

where F_1 and F_2 are arbitrary functions. From condition (4.7), we have

$$z(\tau) = -{}^1/_2\, q_0^2, \qquad \varphi_2(q,\tau) = -{}^1/_2\, q^2 + {}^1/_2\, q + {}^1/_2\, q_0^2,$$
$$\int_{-\infty}^{q_0}\left(\int_{-\infty}^{q_0} w\,dq\right)dq = \chi_1(q_0,\tau) = -\frac{1}{2}x^{-3/2}(x-1). \qquad (4.8)$$

From here we find

$$F_1 = -280\eta,$$
$$\chi_1 = \int_{-\infty}^{q}\left(\int_{-\infty}^{q} w\,dq\right)dq = -\frac{(\omega + 2\eta)}{2x^{1/2}\omega^{1/3}}(\omega - 3\eta)^{4/3},$$
$$\varphi_1 = -\frac{(x-1)(2\eta + \omega)}{4x^{3/2}\omega^{7/3}}(\omega - 3\eta)^{4/3} +$$
$$+ \frac{(\eta + \omega)(\eta - 2\omega)}{4x^{3/2}\omega^{7/3}}(\omega - 3\eta)^{4/3}. \qquad (4.9)$$

For determining F_2, we have

$$\chi_0^{\nu}(\tau) = \int_{-\infty}^{q_0} \nu dq = \frac{13}{220}\omega^6 - \frac{1}{2}\omega^{1/3}\int_{1/3\omega}^{0} F_2(\eta)(\omega - 3\eta)^{2/3} d\eta, \quad (4.10)$$

or, after simple transformations

$$\int_{1/3\omega}^{0} B(\eta)(\omega - 3\eta)^{2/3} d\eta = 2\omega^{-1/3}[{}^1/_4\ \omega^6 - {}^1/_4\ \omega^4 + 2\omega^2 - \delta_0(\tau)],$$

$$B(\eta) = F_2 + {}^{1547}/_3\ \eta^4. \quad (4.11)$$

5. After substitution of (3.14), the equation for the shock wave front has the form

$$\frac{1}{2}\mu_0(\tau) + \frac{\partial\delta^\circ}{\partial\tau} + \frac{1}{4}y^\circ\frac{\partial\delta^\circ}{\partial y^\circ} - \frac{1}{2}\mu^\circ - \frac{1}{2}\left(\frac{\partial\delta^\circ}{\partial y^\circ}\right)^2 = 0, \quad (5.1)$$

where the function $\mu_0(\tau)$ is determined from the condition at the front, at the point $y^0 = 0$

$$\delta(0, \tau) = 0. \quad (5.2)$$

This function can be determined numerically in the same manner that a similar function was determined in [4]. If the function $\delta_0(\tau)$, related to $\mu_0(\tau)$ by relation (3.14), is approximated within the range $\tau_0 < \tau < \tau_1$ by a polynomial, i.e., if one sets

$$\delta_0 = a_0 + a_1\omega + a_2\omega^2 + \ldots$$

then the functions $\varphi_0(q, \tau)$ and $\chi_0(q, \tau)$ will be expressed by finite formulas which contain the unknown coefficients a_k. These coefficients must be determined in such a way that condition (5.2) is satisfied with the greatest possible accuracy within the given range.

$$\delta(0, \tau) = 0.$$

Because of their bulkiness, the formulas for $\varphi_0(q, \tau)$ and $\chi_0(q, \tau)$ are not presented.

The solution obtained satisfies all the boundary conditions of the problem, except the condition for the continuity of the velocity vector component tangential to the shock front (3.11). Within the class of particular solutions (4.1), this condition can be satisfied only approximately.

REFERENCES

1. O. S. Ryzhov and S. A. Khristianovich, "Nonlinear reflection of weak shock waves," PMM, vol. 22, no. 5, 1958.

2. A. A. Grib, O. S. Ryzhov, and S. A. Khristianovich, "Theory of shortwaves," PMTF, no. 1, 1960.

3. A. A. Grib, A. G. Ryabinin and S. A. Khristianovich, "Reflection of a plane shock wave in water from the free surface," PMM, vol. 20, no. 4, 1956.

4. B. I. Zaslavskii, "Nonlinear interaction between a spherical shock wave generated by the explosion of an immersed charge and the free surface of water," PMTF, no. 4, 1964.

5. L. D. Landau and E. M. Lifshitz, Mechanics of Continuous Media [in Russian], Gostekhizdat, Moscow, 1953.

14 May 1968 Moscow